WATER RESOURCES RESEARCH, VOL. 25, NO. 6, PAGES 1187–1194, JUNE 1989

Wetting Front Instability
1. Theoretical Discussion and Dimensional Analysis

R. J. Glass,[1] J-Y. Parlange, and T. S. Steenhuis

Department of Agricultural and Biological Engineering, Cornell University, Ithaca, New York

Highly nonuniform moisture and solute transport in sandy soils has long been noted. A mechanism for nonuniform transport is an instability in the infiltration flow itself, or wetting front instability. When a wetting front becomes unstable, "fingers" form and move down through the vadose zone to the phreatic surface bypassing much of the unsaturated porous medium. In this paper a theoretical framework for wetting-front-instability experimentation is developed through classical dimensional analysis. Relationships between system parameters and initial/boundary conditions and unstable flow field behavior, as characterized by finger width and finger velocity, are derived. The scaling theory of E. E. Miller and R. D. Miller (1956) is used to generalize these relationships to similar porous media with different mean grain sizes and show the effects of coarseness and fluid properties. Other formulations for finger width, derived through linear stability analysis, are examined to arrive at possible functional forms not yielded by dimensional analysis.

Introduction

The stability of an interface between two fluids in a porous medium or in a Hele-Shaw cell across which a pressure gradient is imposed has received continued attention from the 1950s [e.g., *van Meurs*, 1957; *Saffman and Taylor*, 1958; *Chouke et al.*, 1959] through the present [e.g., *Stokes et al.*, 1986; *Saffman*, 1986; *Homsy*, 1987; *Tanveer*, 1987*a*, *b*] due to its importance in the extraction of oil from oil-bearing porous media. A special case of this problem is the stability of the interface that forms when water infiltrates vertically downward into an air-filled or unsaturated porous medium. Here the interface is the well known wetting front that forms due to the nonlinear dependence of the hydraulic properties with moisture content (or pressure). Wetting front instability is driven by gravity and stabilized by surface tension which arises through capillarity within the porous media. While conceivably quite important, wetting front instability has received only sporadic attention as a mechanism for nonuniform water and solute transport.

Analysis of wetting front instability is difficult due to the highly nonlinear behavior of the governing equation [e.g., Richards' equation) for unsaturated flow through porous media. Extension of *Saffman and Taylor*'s [1958] and *Chouke et al.*'s [1959] analyses, however, has predicted instability for an increase, continuous or not, in the saturated hydraulic conductivity with depth; unsaturated infiltration; an increase in the initial moisture content with depth; and wetting front movement from a wetting horizon into a less wetting horizon [*Hill and Parlange*, 1972; *Raats*, 1973; *Philip*, 1975; *Tamai et al.*, 1987; *Hillel*, 1987]. These conditions are extremely widespread. The condition of an increase in conductivity with depth alone is present along much of the eastern seaboard of the United States, in river valleys of postglaciated regions, within the layered tuff deposits at the proposed high level nuclear waste repository at Yucca Mountain, Nevada, and elsewhere. While observation of fingering in the field using traditional field measurement techniques is extremely unlikely, the infiltration of water containing visible dyes as tracers followed by careful excavation of the site has shown fingering to occur in Connecticut [*Starr et al.*, 1978; *Starr et al.*, 1986] and on eastern Long Island, New York [*Glass et al.*, 1988]. *Van Ommen et al.* [1988], *Hendrickx et al.* [1988], and *Van Ommen and Dijksma* [1988] have observed fingers in the Netherlands using a new visualization technique they developed that uses iodide as a tracer. Upon excavation, exposed faces of soil are treated to turn areas containing the tracer blue.

Fingering was noted in a number of early laboratory experiments [e.g., *Tabuchi*, 1961; *Miller and Gardner*, 1962; *Peck*, 1965; *Smith*, 1967] but it was not related to its cause: instability. There have been five reported laboratory studies specifically designed to examine wetting front instability; *Hill and Parlange* [1972] for layered soils, *White et al.* [1976] for air pressure increase in front of the wetting front, *Diment and Watson* [1985] for both layered soils and redistribution following infiltration under different uniform initial moisture contents, *Glass* [1985] for layered sands and repeated infiltration events in layered sands, and *Tamai et al.* [1987] for redistribution following infiltration in glass beads.

The development of the unstable flow field in time and over sequential infiltration events is quite complex. As was demonstrated first by *Hill and Parlange* [1972], for dry initial conditions, fingers have a distinct moisture content structure consisting of a wet core surrounded by dry media (see Figures 1*a* and 1*b*). For these conditions *Glass* [1985] observed the width and velocity of finger propagation in initially dry porous media to be affected by the flow rate through the finger. The higher the flow rate, the wider and faster the finger. With continued steady infiltration for long periods after fingers reach the bottom of the chamber, Glass documented a slow sideways flow of moisture from finger core areas into the dry sand on either side until the chamber is entirely wet. The laterally moving wetting fronts, however, are at a much lower moisture content than the finger core areas, and, after lateral movement subsides, finger core areas persist as conduits for the majority of the flow (Figure 1c). With subsequent infiltration cycles, it is found that flow

[1]Now at Geoscience Analysis Division, Sandia National Laboratories, Albuquerque, New Mexico.

Paper number 89WR00269.
0043-1397/89/89WR-00269$05.00

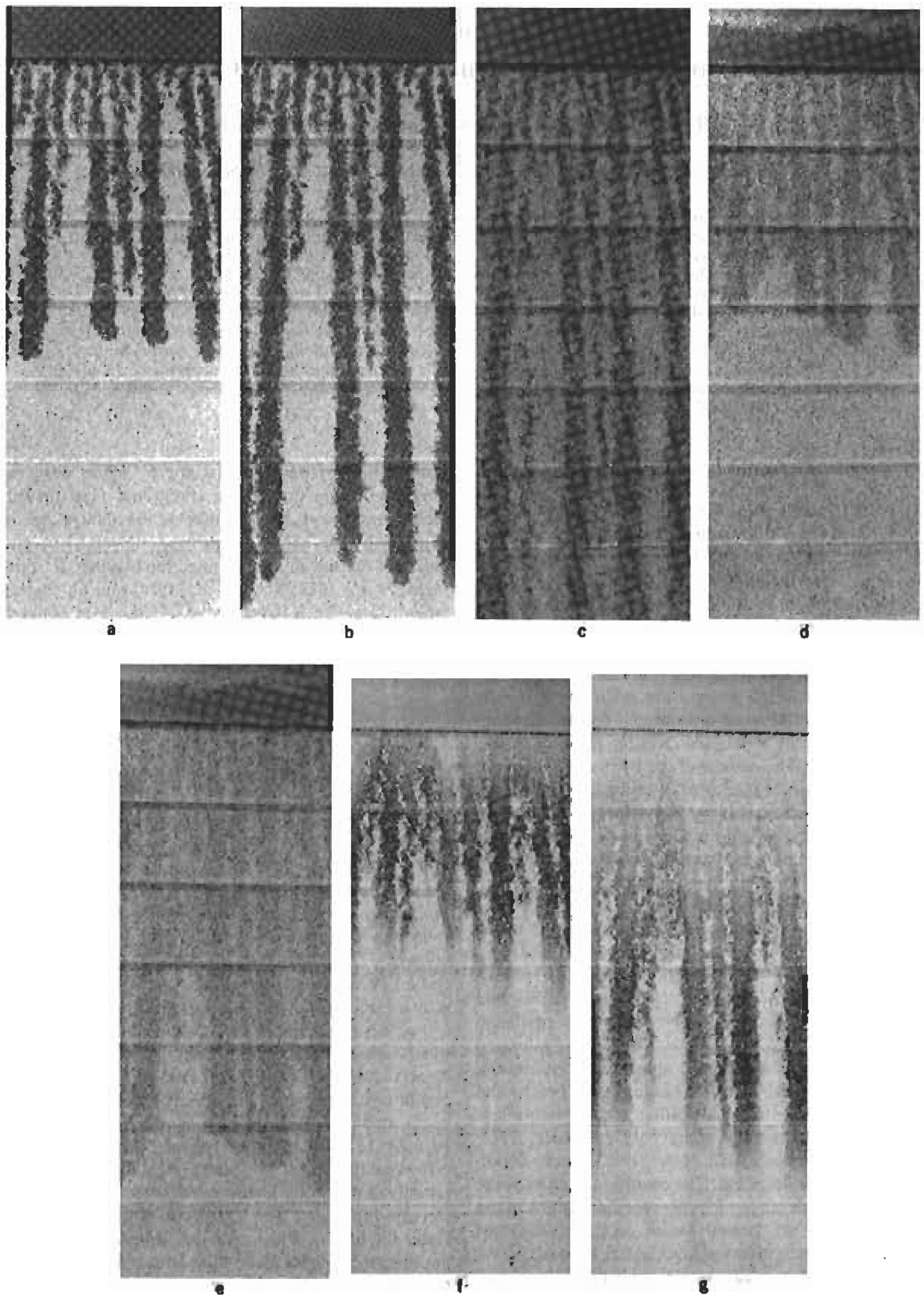

Fig. 1. A sequence of photographs of an experiment of *Glass* [1985] demonstrating finger persistence in a fine- over coarse-layered sand system. The vertical chamber is 30 × 80 × 1 cm with the fine layer at the top of the photograph. Figures 1*a* and 1*b* show the downward movement of fingers in the initially dry bottom layer. In Figure 3c, after 24 hours of steady infiltration, while the chamber is everywhere wet, flow remains concentrated in the original core areas. In a subsequent infiltration event, Figures 1*d* and 1*e* demonstrate the continued persistence of the original fingers. The identity of the core areas between the two cycles is further emphasized in Figures 1*f* and 1*g* which show the movement of a dye pulse through the chamber in the second infiltration cycle.

remains concentrated in the same core areas where fingers first formed thus demonstrating the persistence of fingers from one infiltration cycle to the next (Figures 1*d* through 1*g*). If the experimental chamber is then artificially saturated and drained to field capacity, another infiltration cycle shows the core areas to disappear. Infiltration into this uniform moisture content field near field capacity appears instead to be unstable at a wavelength an order of magnitude longer than in the initially dry sand (on the order of the chamber width). This result is also indicated by the study of *Diment and Watson* [1985] in sands wet below field capacity. The numerical study of *Diment and Watson* [1983], however, indicates that infiltration into uniform moisture content fields of above 0.05 ml/ml are stable.

The purpose of the current research into the phenomenon is to further clarify the wetting front instability process through systematic experimentation. Two questions must be addressed: (1) Under what circumstances does wetting front instability occur, and (2) How can the behavior of an unstable flow field be described? Of particular interest are the properties of fully formed fingers, as they quickly dominate the unstable flow field, persist from one infiltration cycle to the next and influence solute transport greatly.

In this paper, dimensional analysis provides the theoretical framework for the experiments carried out in paper 2 [*Glass et al.*, this issue]. We begin by listing and briefly discussing the system parameters. Dimensional analysis is applied at two distinct scales, the "finger scale" and the "chamber scale." At the finger scale, formulations are derived for finger size (width or diameter) and velocity as functions of porous media properties and initial/boundary conditions. The scaling theory of *Miller and Miller* [1956] is used to predict the role of surface tension, contact angle, fluid viscosity, and mean grain diameter in determining finger size and velocity. At the larger chamber scale, formulations for finger size and velocity are again derived. The formulation for finger size at the chamber scale is compared with those found through linear stability analysis by *Philip* [1975] as corrected by *White et al.* [1976] and by *Parlange and Hill* [1976].

UNSTABLE FLOW FIELD BEHAVIOR AND SYSTEM PARAMETERS

Unstable flow field behavior is to be described as a function of composite system variables and the initial/boundary conditions. Observable variables that may be used to characterize the behavior of the unstable flow field include the number of fingers and their cross-sectional area (or width in two-dimensional systems), average moisture content, pattern of spacing and propagation velocity, as well as the flow into and through individual fingers. Since the preliminary work of *Glass* [1985] found finger width, propagation velocity and the flow rate into and through an individual, fully developed, isolated finger not to vary with depth, these observables are considered properties of a given isolated finger.

The system parameters are given in the appendix. The first two groups of parameters, physical properties of the porous medium and fluid properties, affect their composites in the third group, the conductivity $K(\theta)$ with units of $[L/T]$, fluid pressure potential characteristic relation $\psi(\theta)[L]$, diffusivity $D(\theta)[L^2/T]$, and an integral capillary property, the sorptivity $S[L/T^{1/2}]$, all of which are functions of the moisture content $\theta[L^3/L^3]$. In order to vary the composite properties of porous media systematically, the parameters in the first two groups are varied.

The fourth group, heterogeneity in hydraulic properties, can influence both the stability of a system and the unstable flow field behavior. This influence can be seen clearly in the photographs presented by *Hill and Parlange* [1972] and *White et al.* [1976]. *Diment and Watson*'s [1985] study with initially wet and initially dry sand used different packing procedures for the different moisture contents that may have resulted in different heterogeneity structure. It is conceivable that the type, "intensity," and length scale of heterogeneity could be systematically varied and its influence on wetting front instability characterized. Experimentation in the absence of heterogeneity, however, is crucial for understanding the effect of the other system variables and is of primary concern here.

As in Rayleigh-Taylor instability, the occurrence and dimensionality of fingering is linked to the macroscopic scales of the flow field (the fifth group). The phenomenon can be forced to be two dimensional by constraining the flow field in one of the horizontal dimensions to be less than the dimension of the smallest of the fingers. A one-dimensional flow field can be forced if the second horizontal dimension is likewise constrained. The width of fingers is influenced by the properties of the layer in which they form and by the initial and boundary conditions. Just because fingers do not form at a given scale does not imply that fingers cannot form at a larger scale, i.e., in a wider chamber. Thus chamber width must be considered as a stability parameter.

Parameters in the sixth group describe initial/boundary conditions. In nature, nonuniform initial moisture content is probably the rule rather than the exception as is nonuniform flux or pressure at the surface both in time and space. The air pressure in front of the wetting front is another crucial stability parameter as it influences the infiltration rate. As discussed above with respect to heterogeneity of hydraulic properties, experimentation in uniform initial moisture content fields, with uniform flux at the surface in both time and space without the buildup of air pressure in front of the wetting front is a necessary first step in clarifying the fingering process.

DIMENSIONAL ANALYSIS APPLIED AT THE FINGER SCALE

Dimensional analysis will be used to help both design and interpret experimentation in wetting front instability. Fully formed fingers are studied and their properties of width and propagation velocity are related to initial/boundary conditions and properties of the porous medium in which they form.

Consider a fully formed finger moving through a semi-infinite, homogeneous, isotropic porous medium (Figure 2). The top boundary condition may be characterized by the average flux into the finger, q_F $[L/T]$, given by

$$q_F = Q_F/A_F \tag{1}$$

where Q_F is the flow into the finger $[L^3/T]$ and A_F is the finger cross-sectional area $[L^2]$. We will scale q_F by the maximum flux possible under unit gradient, given by the saturated conductivity, K_s, yielding the dimensionless flux-conductivity ratio R_F

$$R_F = q_F/K_s \tag{2}$$

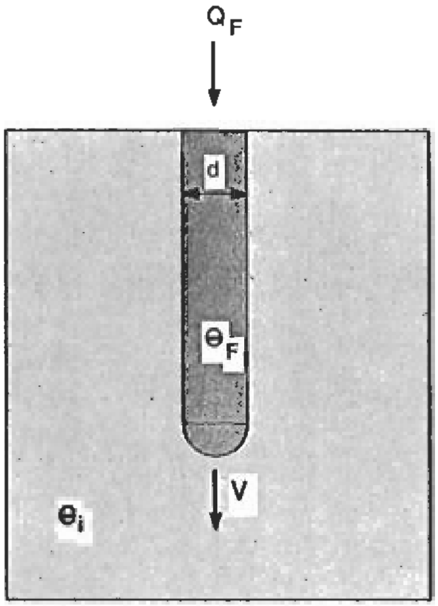

Fig. 2. Drawing of an idealized two-dimensional finger in a semi-infinite isotropic homogeneous porous medium. Initial moisture content is θ_i. The finger has an average moisture content, θ_f, and width d, propagating downward at a velocity v. The flow rate into the finger is Q_F.

The initial condition will be taken as uniform moisture content, θ_i, and be incorporated into the parameter $(\theta_s - \theta_i)$, where θ_s is the saturated value of the moisture content. Here $(\theta_s - \theta_i)$ represents the maximum volume of pores which may be filled.

For the properties of the porous medium, we choose from the composite properties of $K(\theta)$, $\psi(\theta)$, and $D(\theta)$. The choice of properties and dimensionless groups is guided by the nondimensionalization of the flow equation.

Nondimensionalization of Richards' Equation

Flow through a rigid, nonreactive, porous medium is assumed to be governed by the Richards' equation [*Richards*, 1931]

$$\partial\theta/\partial t = \nabla \cdot [K(\theta)\nabla\psi] - \partial K(\theta)/\partial z \tag{3}$$

where t is time $[T]$ and $z[L]$ is taken positive downward. Richards' equation has been demonstrated to model well the infiltration of water into unsaturated porous media for arbitrary initial moisture contents including dry conditions [e.g., *Brooks and Corey*, 1964]. Since finger tip movement is a wetting process and water is being supplied at the top of the layer to the finger under suction, the maximum value of ψ is denoted ψ_{we}, an effective water entry value for the layer. The diffusion form of (3) may be written

$$\partial\theta/\partial t = \nabla \cdot [D(\theta)\nabla\theta] - \partial K(\theta)/\partial z \tag{4}$$

for $D(\theta) = K(\theta)\ d\psi(\theta)/d\theta$. The first term on the right-hand side of each of (3) and (4) embodies movement due to capillarity and the second movement due to gravity. While either form of Richards' equation may be nondimensionalized, we choose to work with (4).

Each of the dimensional variables in (4) is nondimensionalized by a characteristic value. The characteristic values taken should embody those that occur in the problem. The finger width (or diameter), d, is taken as the characteristic macroscopic length scale. For a finger moving down through homogeneous unsaturated porous media, the conductivity may vary from $K(\theta_i)$ to $K(\theta_F)$, where θ_F is the average moisture content within the finger core which may be close to the saturated value θ_s. The pressure head may range from $\psi(\theta_i)$ to $\psi(\theta_F)$, where $\psi(\theta_F)$ may be as high as ψ_{we} and the diffusivity from $D(\psi_i)$ to $D(\psi_{we})$ which may be very large. Since the nonlinear shapes of the forms of $K(\theta)$ and $D(\theta)$ cause most of the unsaturated flow behavior to be concentrated at the high end of θ, we take the characteristic moisture content as θ_F.

The characteristic value of the conductivity is taken as K_F where K_F is $K(\theta_F)$. K_F parameterizes the effect of gravity. To represent capillarity, we choose the sorptivity, S, as it is the relevant integral property of the porous medium that parameterizes the movement of water due to capillary action in the absence of gravity. In addition, S is well defined compared to the soil water diffusivity. The approximate formula of *Parlange* [1975] gives S as

$$S_F^2 = \int_{\theta_i}^{\theta_F} (\theta + \theta_F - 2\theta_i)\ D\ d\theta \tag{5}$$

The subscript F denotes that S is evaluated for the upper bound of the integral in (5) of θ_F, the moisture content within the finger core. The sorptivity is scaled by $(\theta_F - \theta_i)$, since the volume of porous media enters in D and the volume of water in S. So we use $S_F^2/(\theta_F - \theta_i)^2$ to obtain a dimensionless diffusivity.

Dimensionless variables are now defined and denoted by a subscripted star

$$\theta_* = (\theta - \theta_i)/(\theta_F - \theta_i) \tag{6a}$$

$$\nabla_* = d\nabla \tag{6b}$$

$$K_*(\theta_*) = K(\theta_*)/K_F \tag{6c}$$

$$D_*(\theta_*) = D(\theta_*)(\theta_F - \theta_i)^2/S_F^2 \tag{6d}$$

$$t_* = tS_F^2/[d^2(\theta_F - \theta_i)] \tag{6e}$$

The nondimensional form of (4) then is

$$\partial\theta_*/\partial t_* = \nabla_* \cdot D_*(\theta_*)\nabla_*\theta_* - N\partial K_*(\theta_*)/\partial z_* \tag{7}$$

where N is a dimensionless parameter that weighs the effect of gravity to that of capillarity given by

$$N = K_F(\theta_F - \theta_i)\ d/S_F^2 \tag{8}$$

N is a function of initial moisture content, θ_i. Note that θ_i also affects S_F^2 through the lower limit of integration (equation (5)). In addition, θ_i may also effect θ_F, and thus K_F, due to air entrapment.

Since we do not know θ_F a priori, we wish to remove it from (8) in order to estimate N. Experimental evidence indicates that θ_F is approximately equal to θ_s [*Hill and Parlange*, 1972; *Glass*, 1985]. Since $K_F(\theta_F - \theta_i)/S_F^2$ is a weak function of θ_F for θ near θ_F, we will replace θ_F by θ_s to estimate N in (8). Thus we approximate N by

$$N = K_s d(\theta_s - \theta_i)/S^2 \tag{9}$$

where the upper limit defining S is now θ_s. Note that this S would be the value obtained from a horizontal diffusion experiment where θ at the supply side would be θ_s but the pressure there would be ψ_{we}. Thus the value of S we use is different from the traditional value S_0 obtained from the same experiment with the supply pressure at zero.

Since wetting front instability is driven by gravity and stabilized by capillarity, N is a natural choice for a dimensionless grouping of the relevant system parameters and will be referred to as the gravity-capillarity ratio.

Finger Width

Finger width d will be considered a function of characteristic values of the relevant system parameters K_s and S, and initial/boundary conditions θ_i and q_F. Since two dimensions are represented, L and T, three independent dimensionless groups can be formed. The first dimensionless group is $(\theta_s - \theta_i)$. This group serves as a conversion factor between volume (or length) of water and volume (or length) of porous medium. The second dimensionless group is R_F, the flux-conductivity ratio. The third group originates from the nondimensionalization of Richards' equation and is the gravity-capillarity ratio, N. Following the formalism of dimensional analysis, N and R_F are related

$$N = f_{dF}(R_F) \tag{10}$$

where f_{dF} is an, as yet, undetermined function of R_F. Substituting for N and solving for d gives

$$d = \frac{S^2}{(\theta_s - \theta_i)K_s} f_{dF}(R_F) \tag{11}$$

The form of f_{dF} must be determined through experimentation or derived through analysis. As stated earlier, due to the highly nonlinear nature of the phenomenon, analysis is expected to yield only approximations and careful experimentation is needed to properly describe the function f_{dF}.

If $f_{dF}(0)$ is finite, then d should go to a constant given by $f_{dF}(0)S^2/[(\theta_s - \theta_i)K_s]$. Obviously, $f_{dF}(0)$ means the limit of f_{dF} for $R_F > 0$ but very small. As R_F approaches 1, d becomes infinitely large, since the flow is necessarily one dimensional. Here f_{dF} may also be expected to have a slightly different functional form in two and three dimensions where d in two-dimensions would correspond to the width of a finger while in three dimensions d would correspond to the diameter of a cylindrical finger.

Finger Velocity

Finger propagation velocity v will depend on the same system parameters as finger width. To incorporate v, a dimensionless velocity-conductivity ratio, R_{VF}, is defined as

$$R_{VF} = (\theta_s - \theta_i)v/K_s \tag{12}$$

R_{VF} should be a function of the dimensionless groups N and R_F but since N is a function of R_F, R_{VF} is a function of only R_F

$$R_{VF} = f_{vF}(R_F) \tag{13}$$

Here f_{vF} is again an as yet undetermined function of R_F. Substituting for R_{VF} and solving for v gives

$$v = \frac{K_s}{(\theta_s - \theta_i)} f_{vF}(R_F) \tag{14}$$

As before, f_{vF} must be determined primarily through experimentation.

Application to Similar Sands

Equations (11) and (14) for finger width and velocity respectively may be extended through the concept of similar porous media flow systems. This study shows explicitly the effect of the mean grain size and fluid surface tension, viscosity, and density on finger width and velocity for a family of similar porous media. In terms of (7), the concept of similarity between two porous media flow systems demands that $K_*(\theta_*)$, $\psi_*(\theta_*)$, and thus $D_*(\theta_*)$ be identical functions with their scale factors incorporated into N, the gravity-capillarity ratio. Therefore relationships derived between N and R_F from dimensional analysis are expected to apply to two porous media where $K_*(\theta_*)$, $\psi_*(\theta_*)$, and $D_*(\theta_*)$ are the same.

The functional forms of $K_*(\theta_*)$, $\psi_*(\theta_*)$, and $D_*(\theta_*)$ depend on pore-size distribution, shape, and connectivity. Grain size distributions that scale logarithmically and are packed to the same bulk density, are "similar" in the sense of *Miller and Miller* [1956]. Miller and Miller's analysis may be used to scale K_s and S^2 and thus R_F, R_{VF}, and N by

$$K_s = (\rho g m^2/\mu)K_{s*} \tag{15}$$

$$S^2 = (\sigma m/\mu)S_*^2 \tag{16}$$

$$R_F = \mu q_F/(\rho g m^2 K_{s*}) \tag{17}$$

$$R_{VF} = \mu(\theta_s - \theta_i)v/(\rho g m^2 K_{s*}) \tag{18}$$

$$N = \frac{m\rho g}{\sigma} \frac{dK_{s*}(\theta_s - \theta_i)}{S_*^2} \tag{19}$$

where K_{s*} and S_*^2 are dimensionless numbers, ρ is the fluid density $[M/L^3]$, μ is the fluid dynamic viscosity $[M/LT]$, σ is the fluid surface tension $[M/T^2]$, g is the gravitational constant $[L/T^2]$, and m is a microscopic length scale of the porous medium $[L]$ (equal to the geometric mean grain size, for instance). The scaling of N and R_F shows that d will scale by $1/m$ and q_F will scale by m^2 so that dm and q_F/m^2 must be the same for similar flow behavior. Equation (11) may be written more generally as

$$d = \frac{\sigma}{\rho g m} \frac{S_*^2}{K_{s*}(\theta_s - \theta_i)} f_{dF}(R_F) \tag{20}$$

Equation (20) states that for a family of similar sands and q_F chosen such that q_F/m^2 is a constant, the product of d and m is a constant. As m decreases, d increases, i.e., fingers will be wider. The velocity may also be scaled yielding

$$v = \frac{\rho g m^2}{\mu} \frac{K_{s*}}{(\theta_s - \theta_i)} f_{vF}(R_F) \tag{21}$$

Equation (21) shows the velocity to scale by m^2 so that for q_F/m^2 a constant, the ratio of v to m^2 is also a constant.

In the Miller scaled form the effects of σ, ρ, and μ also are seen on finger size and velocity. An increase in the ratio σ/ρ will decrease N and increase the finger width for a given R_F. The ratio μ/ρ which is defined as the kinematic viscosity affects both the variable R_F and R_{VF} such that for m^2R_F or m^2R_{VF} a constant, respectively, $q_F\mu/\rho$ or $v(\theta_s - \theta_i)\mu/\rho$ must be the same, respectively.

Dimensional Analysis Applied at the Chamber Scale

While (11) and (14) apply to individual fingers, often it is the flux through the entire system instead of the flux through the finger that is known. Therefore it is of use to extend the above analysis from the finger scale to the larger chamber scale. We define the system flux-conductivity ratio as

$$R_S = q_S/K_s \tag{22}$$

where q_s is the flux through the system or into the layer $[L/T]$ and K_s is the saturated conductivity within the homogeneous layer of interest. In general, q_S is a function of time and may be calculated given the hydraulic properties between the location of the top boundary condition and the point of interest, the imposed boundary condition and the initial conditions. In the case of a two-layer system with a fine layer overlying a coarse layer, the flux through the top layer becomes a constant once the wetting front has passed into the bottom coarser layer and fingers have formed [*Hill and Parlange*, 1972].

Writing the average gravity-capillarity ratio, $\bar{N}$, for the average finger width, $\bar{d}$, and the average velocity-conductivity ratio $\overline{R_{VS}}$ for the average finger velocity, $\bar{v}$, we have

$$\bar{N} = f_{dS}(R_S) \tag{23}$$

$$\overline{R_{VS}} = f_{vS}(R_S) \tag{24}$$

where f_{dS} and f_{vS} have yet to be determined. Thus

$$\bar{d} = \frac{S^2}{K_s(\theta_s - \theta_i)} f_{dS}(R_S) \tag{25}$$

$$\bar{v} = \frac{K_s}{(\theta_s - \theta_i)} f_{vS}(R_S) \tag{26}$$

and the Miller scaled form of (25) and (26) are

$$\bar{d} = \frac{\sigma}{\rho g m} \frac{S_*^2}{K_{s*}(\theta_s - \theta_i)} f_{dS}(R_S) \tag{27}$$

$$\bar{v} = \frac{\rho g m^2}{\mu} \frac{K_{s*}}{(\theta_s - \theta_i)} f_{vS}(R_S) \tag{28}$$

where R_S is scaled as

$$R_S = \mu q_s/\rho g m^2 K_{s*} \tag{29}$$

R_S can be related to $\overline{R_F}$, now defined using the average flux through the fingers that form in the chamber, $\overline{q_F}$, by

$$\overline{R_F} = \beta R_S \tag{30}$$

β is defined as $A_S/n\overline{A_F}$ *where* n is the number of fingers, $\overline{A_F}$ is the average cross-sectional area of the fingers (or width in two dimensions), and A_S is the cross-sectional area of the system (or width in two dimensional flow). Note that β is a function of $\overline{R_F}$, as shown in paper 2.

In terms of a two-layer system, R_S essentially characterizes the perturbation level due to the change in conductivity across the textural interface. As we show in the next section, relationships derived from linear stability analysis for $\bar{d}$ may readily be written as functions of R_S and thus give insight into the possible form of $f_{dS}(R_S)$.

Relationship Between Formulations Derived Through Dimensional Analysis and the Results of Linear Stability Analysis

An average finger width or diameter is often obtained through linear stability analysis by finding the most rapidly growing wavelength [*Drazin and Reid*, 1981]. The studies of both *Philip* [1975] and *Parlange and Hill* [1976] obtained finger widths in this way and may be manipulated into the form of (25). Linear stability analysis, however, is only valid for infinitesimal disturbances. There is no guarantee that the wavelengths that initially grow the most rapidly end up giving the dominant finger width, especially in a highly nonlinear system.

Philip [1975] derived an equation for finger diameter using the Green and Ampt model of infiltration. While Philip's equation lacks reference to the properties of the porous medium, it does yield a relation for f_{dS}. His equation (30) may be rewritten for the expected two dimensional finger width d_{PH} as

$$d_{PH} = \pi[3\sigma/(\rho g J)]^{1/2} \tag{31}$$

where J is the pressure gradient at the wetting front given by the Green and Ampt equation. While this equation should apply to Hele-Shaw cells, it does not apply directly for porous media. *White et al.* [1976] corrected Philip's equation for flow in porous media to d_w

$$d_w = \pi\left[\frac{3\sigma}{\rho g J(\theta_s - \theta_i)}\right]^{1/2} \tag{32}$$

For q_s constant (32) may be rewritten in the form of (25) by noting

$$J = 1 - q_s/K_s = 1 - R_S \tag{33}$$

to give

$$d_w = \pi\left[\frac{3\sigma}{\rho g(\theta_s - \theta_i)(1 - R_s)}\right]^{1/2} \tag{34}$$

The analytical study of *Parlange and Hill* [1976] uses a relationship between the velocity of a sharp front and that of a diffuse front (due to capillary forces) to calculate the width of a finger assuming the finger is saturated. They derived the following approximation (their equation (19)) for the width of a two-dimensional finger, d_P:

$$d_P = \frac{\pi S^2}{(\theta_s - \theta_i)[K_s - V(\theta_s - \theta_i)]} \tag{35}$$

Here V is the velocity of the wetting front if it were stable. This equation may be easily written in terms of R_S, since the flux through the system is $V(\theta_s - \theta_i)$, yielding

$$d_P = \frac{\pi S^2}{K_s(\theta_s - \theta_i)(1 - R_S)} \tag{36}$$

Equation (36) is in the form of (25).

Comparison of (25) with the equations obtained from *White et al.*'s [1976] and *Parlange and Hill*'s [1976] analyses yield $f_{dS}(R_S)$ as $f_{dS}(0)(1 - R_S)^n$, where n is $-1/2$ and -1, respectively. For R_S going to zero, we expect $\bar{d}$ to go to a constant given by $f_{dS}(0)S^2/[K_s(\theta_s - \theta_i)]$ which is also shown by Parlange and Hill's form with $f_{dS}(0)$ as π. In the case of White et al.'s form this constant would be $\pi[3\sigma/\rho g$

$(\theta_s - \theta_i)]^{1/2}$, which, apart from θ_s, is independent of the properties of the porous medium.

Discussion

Once $f_{dF}(R_F)$ or $f_{dS}(R_S)$ is known, (11) or (25) may be used very simply to assess the stability of the system. Here d may be calculated for the particular porous medium comprising the system. For d less than the width or diameter of the system, L_S, instability is expected, while for d greater than or equal to the system width, stability is expected. Using (11) and (25),

$$S^2/[(\theta_s - \theta_i)K_s]f_{dF}(R_F) \geq L_S \text{ or } S^2/[(\theta_s - \theta_i)K_s]f_{dS}(R_S) \geq L_S \quad (37)$$

is the condition for stability.

The Miller scaled forms of the equations for d and v are of great use in generalizing the analysis, and also offer an efficient route for experimental verification of the formulations obtained for d and v. The dependence of d on mean grain size m of the porous medium is such that the coarser the porous medium, the smaller the minimum finger width that can occur. This may explain why fingering has been noted only in coarse sands whose minimum finger size is most likely to be less than the size of the experimental chamber. For fine sands the minimum finger size is probably larger than the size of most experimental chambers. Changes in fluid properties should also affect finger width and velocity. For instance, the addition of chemicals to the water which decrease the surface tension should decrease the size of the fingers.

On the finger scale, finger width should increase as the flow rate through a finger increases. This is supported by the formulas at the system level of both *Parlange and Hill* [1976] and *White et al.* [1976] which predict the width of a finger to increase with an increase in the flux into the system. While the experiments of *Glass* [1985] indicated this trend, the experiments of *Hill and Parlange* [1972] did not. In a change in R_S from 0.075 to 0.1, Hill and Parlange instead found the number of fingers to increase, while the width and velocity of the fingers that formed were essentially the same.

The discrepancy between theory and the results of *Hill and Parlange* [1972] is likely to be a result of the influence of heterogeneities. The theory is developed for isotropic and homogeneous porous media. In the laboratory, isotropy and homogeneity are extremely hard to obtain and verify. In fact, the photos presented by Hill and Parlange show heterogeneities clearly and their effect to cause finger merger near the top of the bottom layer, finger meandering and uneven finger sides. While heterogeneity is an ever present part of the landscape, an experimental study of wetting front instability in isolation from its complicating effect is essential to understand the basic phenomenon.

Conclusion

Through dimensional analysis we have derived expressions for finger width (or diameter) and finger velocity that are functions of porous media properties and initial/boundary conditions. Miller scaling of the expression for finger width predicts the width to increase as a porous medium becomes finer. In addition, both the formulations of *Parlange and Hill* [1976] and *White et al.* [1976] predict finger width to increase as the flow through the system increases. Instability thus depends on both porous media properties and the flow rate through the system. It is not surprising that fingering has been noted in the laboratory primarily in coarse sands where $S^2/K_s(\theta_s - \theta_i)$ will be small, while for fine-textured soils, the finger width may be on the order of 1 m or more.

Given that application of the linear stability theory of *Saffman and Taylor* [1958] and *Chouke et al.* [1959] to wetting front stability has predicted instability for many infiltration situations that naturally occur and the fact that previous experiments have yielded results that contradict the theory for homogeneous soils, the need for well designed experiments followed by the reevaluation of theory is critical. In paper 2 [*Glass et al.*, this issue] we present the results of careful experiments designed and executed with special emphasis on isotropy and homogeneity in porous media properties. Quantification of the relations derived through dimensional analysis between finger width and velocity and system parameters is accomplished for a particular porous medium. These relations are then compared with the formulations derived through linear stability analysis.

Appendix: System Parameters

The physical properties of the porous medium are (1) mean grain size/pore size (microscopic length scale); (2) grain size/pore size distribution about the mean; and (3) distance between grains/bulk density/porosity.

The fluid properties of the porous medium are (1) surface tension; (2) viscosity; (3) density; and (4) contact angle between liquid, gas, and porous media.

The composite hydraulic properties of the porous medium are (1) conductivity, $K(\theta)$; (2) fluid potential characteristic relation, $\psi(\theta)$; (3) diffusivity, $D(\theta)$; and (4) sorptivity, S.

The heterogeneity of hydraulic properties of the porous medium are (1) type; (2) "intensity" or level; and (3) length scale of heterogeneity.

The macroscopic length scale of porous medium flow system includes vertical thickness and horizontal cross section (controls finger dimensionality) three dimensional; forced two dimensional; and forced one dimensional).

The initial/boundary conditions of the porous medium are (1) initial moisture content, θ_i; (2) flux or pressure supplied at top of system; and (3) air pressure ahead of the wetting front.

Acknowledgments. This research was supported in part by the Annual Allotment Program of the U.S. Department of Interior, EPA grant R81-2919-01-0 and the Department of Agricultural and Biological Engineering, Cornell University, Ithaca, New York.

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R. J. Glass, Geoscience Analysis Division 6315, Sandia National Laboratories, Albuquerque, NM 87185.

J-Y. Parlange and T. S. Steenhuis, Department of Agricultural Engineering, Cornell University, Ithaca, NY 14853.

(Received April 7, 1988;
revised January 17, 1989;
accepted February 1, 1989.)