Wetting FRONT Instability is Unsaturated
Porous Media: A Three-Dimensional Study
in Dity Dry Sand

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Abstract. Three-dimensional fingers (formed by gravity-driven, wetting from saturation in unsaturated porous media were explored through laboratory experimentation. Two sets of experiments were conducted using sandy dry sand in large, 1 meter-diameter columns to gain insight into the development of fingers in high-density columns. The thin set consisted of pushing water into a two-layer sample with a 1 cm gap between the layers. Here, a complicated pattern of fingers formed. The thicker set, water was supplied directly to the sample's bottom 3 cm diameter well. In this case, water was supplied directly to the sample's bottom 3 cm diameter well. Here, a complicated pattern of fingers formed. The two-layer sample was more than 3 times thicker than the single layer. The fingers in the thinnest set were large, while those in the thickest set were small. The limited size of the experiment did not provide an opportunity to examine the influence of both the flow through the fingers and the initial properties of the sample. The observations of the sample's unsaturated zone, however, was an important factor in explaining the experimental results.

Key words. Unsaturated flow, instability.

1. Introduction
The surface transport of water and contaminants through the unsaturated and saturated zones has been found to be extremely variable in time and space. To explain the spatial and temporal distribution of solutions in these zones, much emphasis has been placed on spatial variability in porous media. For example, the capacity of an aqueous phase to travel through cracks, animal burrows, or root holds. However, another factor contributing to transport variability may be flow instability.

A wide variety of instabilities and fingering patterns may occur in porous media. Most often the instabilities are driven by variations in gravity or residual forces, or both. In viscosity-driven instability (in a recent review, see Homy, 1987), the displacement of one fluid by another of different viscosity is found to be unstable when the less viscous fluid displaces the more viscous fluid at a sufficiently high displacement flow rate. When this instability occurs (often formed viscous fingering) fingers form, causing the bypass of a large volume of the porous medium. If the two fluids are immiscible, surface tension between the fluids within individual pores is found to have a stabilizing influence when the medium is wettable to the
invasion fluid. For miscible fluids, diffusion and hydrodynamic dispersion at the front play a stabilizing role. Stabilization by these mechanisms restricts the range of wavelengths that are unstable and thus influences stability criteria.

In hydrology, where gravity is often the major driving force, gravity-driven instability has many examples. Any time a more dense fluid overlies a less dense fluid, diffusion and dispersion of which the fluids are everywhere are counterbalanced by convection currents. Examples are ubiquitous including heavy contaminants as in depleted organic solvents released above the water table and as water leached upon fresh water. Again, as in vascularity-driven processes, the instability is restricted to wavelengths above a lower threshold by capillarity and diffusion/dispersion processes operating at the gas-liquid interface for immiscible and miscible fluids, respectively. It is interesting to note that diffusion itself may play a destabilizing role when more than one component that affects fluid density is present as in systems (Green, 1984; Turner, 1983). There, if two or more components diffuse at different rates, gravity-driven instabilities in multi-component systems may occur. Such 'multi-component convection' processes may be important in understanding fluid and contaminant transport and mixing processes in multi-contaminant systems.

A simple situation where gravity-driven instability can cause fingers to form is the downward infiltration of water from a gas-filled, or unsaturated, homogeneous porous medium. The linear stability analysis of Saffman and Taylor (1958) suggests that the liquid-gas interface will be unstable if its velocity is less than the saturated liquid velocity. This conclusion is valid only if the interface constrains unstable wavelengths to be above a minimum threshold dependent on the properties of the medium (Chokshi et al., 1959; Parlane and Hall, 1976). From the point of view of hydrology where horizontal length scales are not restricted, potentially unstable circumstances commonly occur and the instability has been termed 'wetting front instability'.

Wetting front instability has been observed experimentally in both the laboratory (e.g., Tabori, 1961; Miller and Gardner, 1962; Fink, 1967; Hop and Parlane, 1972; White et al., 1976; Glass and Sneath, 1984; Domnesti and Waston, 1985; and Tamis et al., 1987) and in the field (e.g., Spirt et al., 1971; Spirt et al., 1986; Glass, 1986; van Ommen et al., 1988; Hendricks et al., 1988, and van Ommeren and Dijkstra, 1988). Most recently it has been studied experimentally in thin slab chambers (3 cm or approximately 10 mean grain sizes in thickness) that force macroscopic two-dimensional flow in initially dry porous media (Glass et al., 1989a,c). The development of new techniques for visualization of the moisture content within such thin slab chambers has allowed understanding of the basic mechanism of finger persistence (Glass et al., 1989c). While results in two dimensions are important for checking theory and in advancing the understanding of the phenomenon, exploration of wetting front instability in three-dimensional systems is crucial for obtaining results applicable in the field. A three-dimensional study poses experimental difficulties above those in two dimensions. Because fingers in large diameter columns form within the porous medium and

**2. Theory**

Unstable flow field behavior as defined by finger width (or diameter in our case) and finger velocity may be related through dimensional analysis to relevant system parameters (Glass et al., 1989a). Finger diameter, d, as an isotropic and homogeneously porous medium is shown to be a function of the layer properties, the initial moisture content, \( \eta_0 \), and the top boundary condition given as the average flux entering the individual finger from above, \( q_0 \). The value of \( q_0 \) is given by the flow rate through an individual finger, \( q_0 \), divided by the cross-sectional area of the finger, A. The layer properties selected through a non-dimensionalization of the problem are the saturated conductivity, \( K_s \), and the square of the aperture, \( S^2 \), where \( S \) is evaluated between \( \eta_0 \), and the saturated value of the moisture content, \( \eta_s \), corresponding to a potential of \( \psi_0 \), the water entry value. The ratio of \( K_s \) and \( S^2 \) divided by \( (K_s - \eta_s) \), yields a length scale that embodies the ratio of capillary to gravity forces and thus is a relevant scaling factor for the finger diameter. Therefore a dimensionless parameter, the gravity-capillary ratio, \( N_s \), is defined as

\[
N_s = \frac{d}{\sqrt{(K_s - \eta_s)}}
\]

(1)

The average flux through the finger, \( q_f \), divided by \( K_s \) of the layer yields a second dimensionless parameter, \( R_s \), termed the flux-conductivity ratio

\[
R_s = \frac{q_f}{K_s}
\]

(2)

which should also influence finger width. Through dimensional analysis it may be shown that \( N_s \) must be a function of \( R_s \) and \( d \) is found to be

\[
d = d^*R_s^{-1/2}f(d/R_s)
\]

(3)

where \( f \) is a function of \( R_s \) not yielded by dimensional analysis.
To obtain a similar relationship for the finger velocity, \( v \), a third dimensionless parameter, the velocity-conductivity ratio, \( R_v \), is defined

\[
R_v = x \frac{v}{K}
\]

(4)

and because \( N \) and \( R_v \) are related by Equation (3), dimensional analysis shows \( R_v \) to be a function of \( R_s \) as well

\[
R_v = \frac{K}{f_s(R_s)}
\]

(5)

where \( f_s \) is a function of \( R_s \), again not given by dimensional analysis. Equations (3) and (5) are general and are not dependent on the dimensionality of the fingers. The functions \( f_1(R_s) \) and \( f_2(R_s) \), however, may have different forms in two or three dimensions where \( d \) in two dimensions would correspond to the width of a finger while in three dimensions \( d \) would correspond to the diameter of a cylindrical finger. The functions \( f_1(R_s) \) and \( f_2(R_s) \) may be found through either analytical or experimental approaches. Analytical approaches to determine these functions are found on linear stability theory and thus approximate theory (e.g., Philip, 1975; and Parlange and Hill, 1978 for the two-dimensional case). Because it is often difficult to choose the proper approximation when a \( p \)-dependence has not been adequately described experimentally, the experimental approaches are important for gaining analytical development. Such a combination was used for the two-dimensional case to show correspondence between analytical and experimental results for \( f_s(R_s) \) (Glas et al., 1990b).

The stability of a given three-dimensional, vertical infiltration system depends on the expected finger diameter calculated for the layer and the maximum horizontal dimension of the system. If the finger diameter is less than the maximum horizontal system dimension, then instability is expected; otherwise the system is stable. So, for \( N \) calculated with \( R_s \), the maximum horizontal width, substituted for \( d \), Equation (3) yields

\[
N < \frac{K}{R_s}
\]

(6)

for stability. The application of (6), however, is difficult as \( n \) is the flux through the system that is usually known and not the flux through individual fingers. In very homogeneous systems, Glas et al. (1990b) found that for the two-dimensional case, fingers that form with a given flux through the system, \( q_s \), will have very similar flow rates and cross-sectional areas; thus, the average flux through a finger, \( q_f \), and \( q_s \) may be related by

\[
q_f = q_s
\]

(7)

\( \beta \) is defined by \( \lambda_s/\lambda_f \), where \( \lambda_s \) is the number of fingers, \( \lambda_f \) is the average finger cross-sectional area, and \( \lambda_f \) is the cross-sectional area of the system. \( \beta \) is simply the reciprocal of the fraction of the cross-sectional area of the chamber in fingers and

in unsaturated porous media

will itself be a function of \( q_s \) that must be found experimentally. For heterogeneous systems, \( \beta \) may be applied after it has been determined. In more heterogeneous systems, where all fingers may eventually merge into one, \( \beta \) may be better applied with \( q \), substituting for \( q_s \).

Thus \( N, \beta, \) and \( R_s \) predict only both stability and finger properties through the relations in Equations (7), (5), and (6). All these dimensionless parameters will be influenced by initial moisture content through its effect on porous media properties, \( K \), and \( q \) as well as the initial condition \( (K_e, \gamma) \). Using the concept of similarity developed by Miller and Miller (1956), \( N, \beta, \) and \( R_s \) may also be scaled to yield the effects of mean grain size and fluid properties (Glas et al., 1990a). Models that relate mean grain size, grain size distribution, and bulk density to hydraulic properties (e.g., Haverkamp and Parlange, 1980) can be used to demonstrate the effects of these on stability and finger properties as well.

It is also important to note that \( R_s \) and \( R_v \) simply scale the flux through the finger and finger velocity by the maximum value of each that may be attained \( q_s \) a specific system with unit hydraulic head gradient. (Glas et al., 1986b) these values were taken as \( K_e \) and \( K_e (K_e, \gamma) \) respectively, yielding Equations (2) and (4). However, because of other complicating effects such as air entrainment, these values may be different and \( R_s \) and \( R_v \) refined accordingly.

3. Experimental Method

The special experimental methods and procedures developed for the study of writing front instability in two-dimensional slab chambers by Glas et al. (1989b) were adapted for three-dimensional systems in the present study.

3.1. Experimental Sands

The sands used in the experiments were generated from commercially available white silica sand by mechanical sieving. The coarse bottom layer, in which fingers were studied, was composed of 14 to 20 fraction sand (U.S. sieve series, 1.25 mm to 0.7 mm). Finer top layers are also referred to by the same sizes that define their fraction. The sand was cleaned before and between uses with hot soapy water after which it was rinsed thoroughly with tap water followed by distilled water. The sand was dried in an oven at 50°C and, because no effort was made to restrict movement of water vapor from the air into the sand after drying, the initial moisture content, while not zero, was very close to zero. The grain size distribution of the 14-20 sand used in these experiments was skewed toward the fine end compared to the same 14-20 sand used in the two-dimensional study (Glas et al., 1989b) and thus measured \( K_e \) and \( K_e \) were lower.
1.2. EXPERIMENTAL CHAMBER, FILLING AND PACKING

The chamber consisted of twelve 30 cm diameter rings 5 cm in height. The rings were held between two 3 cm square aluminum end plates by four sets of cables equipped with turnbuckles attached to each of the plates which were tightened to hold the rings in place (Figure 1a). Four thin aluminum slings were placed between the rings at each level in line with the four holding cables before tightening. These were to create small cracks between the rings so that air could escape freely and water movement would not be affected by any internal air pressure buildup in the chamber. The top plate had a 30 cm diameter hole corresponding to the chamber diameter.

The chamber was filled through an extension consisting of three 30 cm diameter rings (Figure 6b). The top, middle, and bottom rings were 20, 30, and 10 cm high, respectively. Between the top and middle rings, a piece of perforated sheet metal with approximately 4 mm holes was placed to reduce the sand fill rate. Between the middle and bottom rings and the top plate, two five (4 mm) wire screens were placed to further randomize the falling sand. The filling apparatus was inserted onto top of the chamber and sand was poured onto the top, maintaining the level of sand in the filling extension higher than the perforated screens so that an even distribution of sand was ensured.

When filled with sand, the chamber was packed with a drop impact method using a packing apparatus that filled the chamber 1.25 cm at a time and then dropped it to the ground. After packing, the upper 30 cm of sand was removed as it was found to be of a lower bulk density than below, where the bulk density was a constant reproducible 1.59 g/cm³. For the two-layer experiment, a thin piece of paper towel was placed on top of the bottom layer to ensure a sharp textural interface, and the top layer was added and packed using the traditional tamping method.
To measure the cross-sectional area of fingers that formed in a particular experiment, we counted, in the fact that for coarse sands, the time scale for vertical finger growth is much less than the time scale for horizontal finger widening. When an experiment was sufficiently brief, measurement of finger cross-sectional areas by destructive sampling of the experiment leads only to slight over-estimation of finger dimensions. So that the wet regions could be easily seen, distilled water with a low concentration of USDA Sand #3 (0.0025%) was used in the experiments. The column was cut up radially after an experiment to roughly finger cross-sections with depth, the surface was cleaned with a vacuum to remove sand that had been moved during the slicing process, and finger cross-sectional areas were drawn on acetate sheets. The area of the individual fingers were then measured from the drawings.

All experiments were conducted at a constant 20°C within a controlled environmental chamber. The flow rate through individual fingers was monitored through a ‘drill section’ that consisted the lowest section in the chamber. The drill section was constructed out of an array of 10 cm high, 2.5 cm square tubes braided together into a checkerboard pattern. The tube array was installed on a sheet of PVC that formed the bottom of the chamber area which was glued to a 10 cm high ring of the chamber PVC pipe. At the bottom of each tube a brass nozzle containing a wing of the tube, 10 cm in this case, was higher than the capillary fringe of the sand in the chamber so that movement within the saturated zone that forms at the bottom of the chamber was restricted and the flow entering the tube at the top could be measured. The drill-slip allowed the monitoring of the flow distribution and its evolution in time through the 107 pins (figure 8).

Finger tip velocity was measured experimentally by using the tip as a switch. An electrical potential was placed between the fluid at the top of the chamber and the individual screen at the bottom of each drill section. The electrical field across measure connected to each of the 97 accesses at the bottom was monitored for all of the access ones a second using an analogue-to-digital converter and recorded on an IBM AT. When a finger tip occurred at the bottom of one of the drill sections, the voltage would jump as the circuit was completed.

3.4. EXPERIMENTS CONDUCTED

Two types of experiments were conducted to determine finger velocity and distance in initially dry sand as functions of the dimensionless flux-conductivity ratio, \( R_f \). The first type was conducted, two-layer experiments with a fine-textured layer overlying the coarse 10-20 sand. A series of two-layer experiments were conducted using three different top layers (see Table 1) at a ponding level of 1.5 cm above the top sand layer. Only fingers that were isolated from the other fingers could be used to determine relationships between \( Q, R_f, d, \) and \( v \). Because of the complicated pattern of interaction among the fingers that formed in the two-layer experiments, a goal of only 11 fingers from the two-layer experiments could be considered meaningful and yielded flow rate, velocity, and cross-sectional area measurements. In addition, the flow rate through these fingers spanned only a very narrow range of \( R_f \) (0.02 to 0.07).

To obtain measurements that would give the behavior of \( d \) and \( v \) in time throughout the full range of \( R_f \), from 0 to 1, a second type of experiment was conducted where individual fingers were formed from 2 cm diameter tubes supplied by individual pumps. Each tube was covered by a fine wire mesh and applied directly to the top of the bottom layer containing sand areas of known strength. A comparable method was first used successfully by Saffman and Taylor (1958) in Hele-Shaw cells to study individual fingers and later by Glass et al. (1977) to study two-dimensional fingers in unconsolidated porous media. Thus, the flow rate through the fingers, \( Q_f \), was supplied directly and \( R_f \) spanned the range from 0.18 to 0.94. Thirty-one fingers were simulated in the area source experiments, yielding in combination with the two-layer experiments a total of 42 individual fingers to evaluate the functions of \( R_f \) in Equations (3) and (5).

4. Results

The chamber-scale results of the two-layer experiments are presented in Table 1. Figures 3a and 2b show chamber cross-sectional drawings of wet regions (shaded) with depth for experiments 1 and 3. Figure 2c shows a photograph of one of the 10 cm sections after it had been removed and some of the dry sand blown away. The fingers are seen to be fairly uniform, vertically standing cylinders. In a very homogeneous porous media, finger meander only slightly to the horizontal and thus finger is less important than in more heterogeneous materials. However, we also find that many fingers form in a homogeneous media and in fingers become wider, interaction between fingers cannot be ignored. This is seen in the high flow rate experiment (experiment 3, Figure 2b) where many large fingers next to
each other and it was impossible to find isolated fingers from which to obtain measurements.

Table I suggests, while not conclusively because of the small number of experiments performed, that an increase in flow rate through the system, and thus $R_e$, increases the fraction of the drawdown occupied by fingers and the average $d$ and $v$ of the fingers. The effect on the number of fingers that form. This same result was found earlier for two-dimensional systems (Glass et al., 1980b).

The relationship between finger width, velocity, average moisture content, $Q_e$ and $R_e$, Figures 3, 4, and 5 show the effect of $Q_e$ on the finger velocity, average moisture content, and diameter, respectively. The average moisture content for a finger may be calculated by dividing the flow rate through the finger by the volume wetted per unit time calculated from the average velocity and cross-sectional area of a finger. The plots show that finger velocity, average moisture content, and diameter all are higher when $Q_e$ is higher.

The slope of the finger velocity versus $Q_e$ plot steadily decreases as $Q_e$ increases and is expected to approach asymptotically the saturated pore velocity given by $R_e/v$. (Re - $d$), as was the case for two-dimensional fingers. The relationship between finger velocity and $R_e$ is found to be a straight line throughout the full range of
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Fig. 2c: Photograph of a 10 cm high section removed from the column after the devil was again washed (shown in Figure 2a,b). The photograph was taken from approximately the 4:0'clock position of Figure 2a.

Fig. 3: Finger velocity, \( v \), as a function of the flow rate through the finger, \( Q_r \).

Fig. 4: Finger moisture content, \( \theta_r \), as a function of the flow rate through the finger, \( Q_r \).

\( R_e \), explored in the experiments, with the least squares best fit line given by \( \tau = 45.39R_e + 11.4 \) and an \( r^2 \) coefficient of 0.97. The velocity evaluated at \( R_e = 1 \) gives the value \( v \) approached asymptotically in Figure 3 to be 56.8 cm/min. This value is near but 7% less than the value calculated from measured \( K_r \) (24.7 cm/min) and the porosity (calculated from the average bulk density to be 0.40) of 61.3 cm/min.

The average moisture content shown in Figure 4 increases rapidly at low \( Q_r \), then levels off and, based on the results in two dimensions, should approach \( \theta_r \), which should be near the porosity, at high \( Q_r \). However, the asymptote shown in Figure 4 of 0.30 is 27% lower than expected. In combination with the lower-than-expected maximum finger velocity shown in Figure 3, the results suggest that air entrapment may be removing pores from participation and thus playing a role in the phenomenon. With this in mind, we take the maximum moisture content of a finger from Figure 4 and denote it as \( \theta_r \), with the corresponding maximum conductivity \( \theta_r' = K(a \theta) \). These maximum values now replace \( \theta_r \) and \( K \) in Equations (4) through (7) except where the ratio \( StK(\theta_r - \theta) \) occurs, as this combination is not very sensitive to the moisture content in the finger (Glass et al., 1989a). The factors \( a \) and \( \gamma \) account for the effects of air entrapment and are included with \( \theta_r \) and \( K \), respectively. everywhere the flux through the finger or finger velocity is weighed against the maximum flux \( \gamma K(\theta_r - \theta) \) or velocity \( \gamma K(a \theta_r - \theta) \) under unit gradient, respectively. In the definitions of \( R_e \) and
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\( R_{er} \). Thus \( N \) retains its former definition while \( R_{er} \) and \( R_{fr} \) will henceforth be denoted as \( q_s / \kappa_s \) and \( r_{fr} / \kappa_f = 8 \pi \rho_y \), respectively.

Figure 5 shows finger cross-sectional area to increase linearly with \( Q_s \). This linear relation between finger cross-sectional area, \( A \), and the flow through the finger, given by

\[ A = m_s Q_s + b_s \]  

(8)

may be used to find the form of \( f_R \) in Equation (3). As \( Q_s \) becomes large, \( b_s \) may be neglected and we find \( f_R = Q_s / A \). Because \( f \) approaches \( \alpha \) as \( x \) approaches \( h \), and \( \alpha \) approaches \( n \) for \( Q_s \) large, then \( f_{1/2} \) must approach \( \alpha_k \) for \( Q_s \) large. Thus \( m_s = \gamma_k \) and

\[ A = \frac{Q_s}{\gamma_k} + b_s. \]  

(9)

Writing Equation (9) in terms of the nondimensional finger flux-conductivity ratio \( R_{er} \), gives

\[ \frac{A}{R_{er}} = \frac{b_s}{1 - R_{er}}. \]  

(10)

The finger cross-sectional area may be related to the finger diameter by \( d = 2 \alpha (n / \gamma_k)^{1/2} \) so

\[ d = \frac{2}{\alpha_k^{1/2}} \left( \frac{b_s}{1 - R_{er}} \right)^{1/2}. \]  

(11)

Evaluating (11) and (3) for \( R_{er} \) at 0 yields \( b_s \) as

\[ b_s = \frac{\pi}{4} \left( \left( \frac{d}{2} \right)^2 \right) \times \left( \frac{S}{\psi_w} \right). \]  

(12)

where \( f_{1/2}(0) \) is the value approached by \( f_R \) for \( R_{er} \) small. \( \psi_w \) was measured for the 14-20 sand using the constant head method to be 24.7 cm/min. The value of the square of the sorptivity evaluated between \( \psi \) and \( d \) at a potential of zero, \( \psi_w \), was measured through a vertical infiltration experiment using an adaptation of the method of Takashi (1969) proposed by Parlane et al. (1968). For this coarse sand, the method consisted of experiments to small initial heads (5-10 cm) and short times (1-10 sec) thus decreasing experimental error. The average \( \psi_w \) value measured in six experiments was 64.8 cm²/cm/min with a standard deviation of 16.5. The value of \( S \) with a supply pressure of \( \psi_w \) may be calculated using the approximate formula for \( S_0 \) of Parlane (1975),

\[ S_0 = \frac{d^2}{2 - 2 \kappa \psi_w / \kappa_f}. \]  

(13)

\( \psi_w \) was estimated gravimetrically from a vertical rise experiment to be \( = 3 \) cm.

With \( \psi \) taken to be zero and \( S_0 \) as 0.4, \( S^2 \) is calculated to be 5.5 cm²/min. A least
squares best fit line to the data in Figure 6 gives the slope, \( m_s \), as 0.029 and the intercept, \( b_s \), as 4.10 with an \( R^2 \) coefficient of 0.993. The value for \( \gamma_s \) obtained from the reciprocal of \( m_s \), is 55.9 cm/min which yields \( \gamma_s \) of 0.68. Taking \( \gamma_s \) as zero, \( f_D(0) \) is calculated from Equation (12) to be 4.1. Thus in the form of Equation (3),

\[
d' = 4.1 - \frac{55.9}{X_a(X_a - 6)} (1 - R_s) \gamma_s
\]

A plot of observed \( d' \) versus Equation (14) is shown in Figure 7. It should be noted that while \( b_s \) is given presently by Equation (8), the difficulties of measuring \( S \) in course sands causes the value of \( f_D(0) \) in Equation (12) to be known much less precisely.

The product of finger cross-sectional area and velocity plotted versus the flow rate through a finger is seen in Figure 8 to also be a straight line. This linear relation given by

\[
A'V = m_s Q + b_s
\]

may be used to obtain \( v \) in the form of Equation (4). As in Equation (8), when \( Q_v \) becomes large, \( b_s \) may be neglected and \( Q_v = A'v \), i.e., for large \( Q_v \) the maximum content in the finger approaches \( -b_s \), so that \( Q_v = A'v \). Substitution for \( m_s \) and rearrangement gives

\[
v = \frac{-b_s}{4.1} \frac{A'v}{(a/d) - 4} A'v
\]

Substitution of Equation (10) for \( A \) yields

\[
v = \frac{-b_s}{4.1} \frac{b_s (a/d) - 6)}{b_s (a/d) - 6) \gamma_s}
\]

Written in the form of Equation (5),

\[
v = \frac{-b_s (a/d) - 6)}{b_s (a/d) - 6) \gamma_s}
\]

While it is clear that finger velocity will be zero when \( R_s = 0 \), our data suggest that for \( R_s \) small but positive \( v \) approaches a limit \( v_{mn} \) which is constant. The constant \( b_s \) should be a product of the minimum finger area given by \( b_s \) and \( v_{mn} \) given by

\[
v_{mn} = \frac{C_s}{(a/d) - 6)}
\]

where \( C_s \) is simply a factor that reduces the maximum velocity to \( v_{mn} \). Thus

\[
b_s = b_s C_s \frac{55.9}{(a/d) - 6)}
\]

and substitution into (10) yields

\[
v = \frac{-b_s (a/d) - 6)}{b_s (a/d) - 6) \gamma_s}
\]
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Through experimentation in unsaturated two-dimensional flow fields caused by a fine layer overlaying a coarse layer, Glass et al. (1989b) evaluated the functions \( L_\sigma (R_\sigma) \) and \( L_\sigma (R_\sigma) \) for two-dimensional fingers yielding

\[
d_\sigma = C_{\text{stab}} \frac{R_\sigma}{K_\sigma (R_\sigma - R_\sigma^*)} \left[ \frac{1}{1 - R_\sigma^*} \right]
\]

and

\[
y_\sigma = \frac{R_\sigma}{K_\sigma} C_{\text{stab}} \frac{R_\sigma (1 - C_{\text{stab}})}{R_\sigma^*}
\]

where \( C_{\text{stab}} \) was found to be 3.6 and \( C_{\text{stab}} \) was 0.61 and the subscript \( 2d \) denotes two-dimensional.

The functional form of the dependence of finger width and diameter on \( R_\sigma \) in two and three dimensions is seen to be different by a power of one-half as finger width goes as \( (1 - R_\sigma^*)^{-1/2} \) (Equation 23) while finger diameter goes as \( (1 - R_\sigma^*)^{-1/2} \) (Equation 14). However, because in two-dimensional systems, all definitions are per unit thickness and not per unit area, finger width may be considered as area per unit thickness and so in comparison with Equation (10), the cross-sectional area per unit thickness in two dimensions and the cross-sectional area in three dimensions have the same behavior in \( R_\sigma \).

The behavior of finger velocity with \( R_\sigma \) in two- and three-dimensional systems is seen to have exactly the same form when written with respect to the maximum and minimum finger velocity \( C_2 \) and \( C_{\text{stab}} \) for simply the ratio of the minimum finger velocity to the maximum finger velocity. This ratio and the maximum velocity are different for two- and three-dimensional fingers with air entrainment again playing a role.

With each of Equation (14), Equation (6) may be written to yield the stability condition

\[
L_\sigma < d
\]

As discussed earlier, in order to apply Equation (24) easily, a relation is used for \( d \) as defined in Equation (7) so that Equation (15), when written for \( d \), may be written in terms of \( d_\sigma^* \) in two-dimensional homogeneous systems and \( d_\sigma^* \) would then be expected to be

\[
\beta = \frac{L_\sigma}{R_\sigma^*}
\]

where \( R_\sigma^* \) is defined with \( \beta \). While the two-layer data for the current experiments are too limited to allow an independent determination of a relation for \( \beta \), a value of \( \beta \) calculated from Equation (25) for experiments 2 and 4 shows Equation (25) to be a factor of 3 to 4 too high. Equation (25), however, was found for the two-dimensional case to overestimate \( \beta \) for very small \( R_\sigma \) as it was in experiments 2 and 4 (0.05 and 0.06, respectively). It is clear that additional experimentation with full unsaturated flow fields is required to resolve the formulation for \( \beta \).

References


